# Technical Notes

*TECHNICAL NOTES are short manuscripts describing new developments or important results of a preliminary nature. These Notes cannot exceed 6 manuscript pages and 3 figures; a page of text may be substituted for a figure and vice versa. After informal review by the editors, they may be published within a few months of the date of receipt. Style requirements are the same as for regular contributions (see inside back cover).*

## Modal Coupling in Lightly Damped Structures

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### Introduction

THE linear dynamic analysis of lightly damped structures typically includes an evaluation of normal modes and frequencies. The classical structural dynamics eigenproblem ignores any damping in the structure, so that the normal modes are always real. These modes, or a portion of them, are used to transform equations of motion to modal coordinates, such that the matrix equations are diagonal and uncoupled except for damping. Special forms of damping have been assumed to justify the use of a diagonal modal damping matrix so that the equations of motion are theoretically uncoupled. Proportional damping[1] is one example. Other special cases have been shown to produce a diagonal matrix also.[2] None of them have much physical justification. The purpose of this Note is to show that the equations of motion are dynamically uncoupled even if the modal damping matrix is not diagonal, as long as frequency separation among the modes is adequate. In any case, coupling will involve only the closely spaced modes. Separation criteria depend on the amount of damping in the structure.

### Scaling Transformation

To demonstrate the foregoing assertion, one may consider equations of motion in the following matrix form

$$m\ddot{x}(t)+c\dot{x}(t)+kx(t)=P_x g(t) \tag{1}$$

where $m$, $c$, and $k$ denote square mass, damping, and stiffness matrices; $x(t)$ is a vector of structural displacements; $P_x$ is a force distribution vector; and $g(t)$ is a scalar function of time. If the modal transformation is given by $x=\phi q$, where modal coordinates are denoted by the vector $q$ and $\phi$ is the modal matrix, then

$$[(\omega^2-\Omega^2 I)+i\Omega\xi]H_q(i\Omega)=\phi^T P_x = P_q \tag{2}$$

where

$$i=(-1)^{1/2}$$

$\phi^T m\phi = I$ (identity matrix)

$\phi^T c\phi=\xi$ (modal damping matrix)

$\phi^T k\phi=\omega^2$ (diagonal matrix of eigenvalues)

and $H_q(i\Omega)$ is the complex frequency response vector, i.e., the ratio of response to input, transformed to the frequency domain. The matrix within square brackets on the left-hand side of (2) is the complex modal impedance matrix and will be denoted by $Z(i\Omega)$.

Following a derivation in Refs. 3 and 4, the modal damping matrix may be written as the sum of a diagonal matrix $\xi_d$ and its complement, $\xi_n \equiv \xi - \xi_d$. That is, $\xi=\xi_d+\xi_n$. Similarly,

$$Z(i\Omega)=Z_d(i\Omega)+Z_n(i\Omega) \tag{3}$$

where

$$Z_d(i\Omega)=[(\omega^2-\Omega^2 I)+i\Omega\xi_d] \tag{4}$$

$$Z_n(i\Omega)=i\Omega\xi_n \tag{5}$$

Finally, a complex scaling transformation $D$ may be defined such that $q=D\gamma$, and $D=Z_d^{-1/2}(i\Omega)$. Substition of (3) into (2) and transformation to $\gamma$ coordinates yields

$$[I+\tilde{Z}_n(i\Omega)]H_\gamma(i\Omega)=P_\gamma \tag{6}$$

where

$$\tilde{Z}_n(i\Omega)=Z_d^{-1/2}(i\Omega)Z_n(i\Omega)Z_d^{-1/2}(i\Omega) \tag{7a}$$

$$H_\gamma(i\Omega)=Z_d^{-1/2}(i\Omega)H_q(i\Omega) \tag{7b}$$

$$P_\gamma(i\Omega)=Z_d^{-1/2}(i\Omega)P_q \tag{7c}$$

By definition, $\tilde{Z}_n(i\Omega)$ will have all zeros on its diagonal. The magnitude of each off-diagonal element provides a direct measure of the degree of coupling between the corresponding two modes.

Figure 1 helps to illustrate the point. If $\xi_n$ is assumed for simplicity to have off-diagonal elements which are all equal, say unity, then values of $Z_n(i\Omega)$ may be pictured as points lying on the surface which is plotted against structural frequencies $\omega_j$ and $\omega_k$ representing the $j$th and $k$th modal frequencies of the structure. The surface looks like a moun-

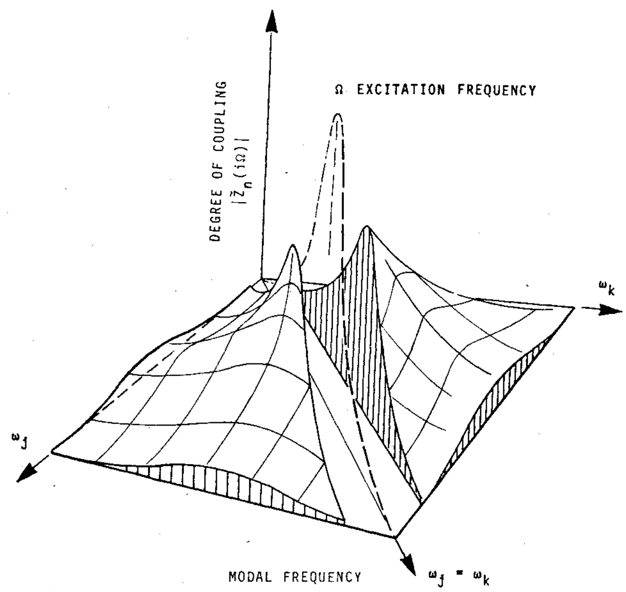


**Fig. 1 Illustration of modal coupling.**

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tain with a slice removed from its center. The mountain will have unit height before removing the slice, whenever the modal damping ratios, $\zeta_j$ and $\zeta_k$ are equal. The width of the slice, measured parallel to either axis, is equal to the difference between frequencies of two adjacent modes. The further apart adjacent modes are, the broader the slice and the lower the portion of the mountain remaining. The degree of modal coupling corresponds to the height of the remaining portion. From this illustration it is apparent that for widely separated modes there is negligible modal coupling *even when the off-diagonal terms of* $\xi$ *are not small compared to the diagonal*

Intuitively, one may reason that as long as damping forces in the structure are small compared to inertia and stiffness forces (light damping) they will be important only to the extent that they dissipate energy over a period of time, and will not couple structural modes which are well-separated in frequency because of high cross-modal impedance. On the other hand, when two or more modal frequencies are closely spaced, cross-modal impedance is low. Inertia and stiffness forces tend to be balanced so that small damping forces are comparable to the small differences between inertia or stiffness forces.

## Modal Separation Criteria

One may neglect coupling between two modes whenever the corresponding element of $\tilde{Z}_n(i\Omega)$ is small compared to unity. This condition may be stated, dropping the argument $i\Omega$ from $\tilde{Z}_n(i\Omega)$,

$$|e_k^T \tilde{Z}_n e_j| << 1 \tag{8}$$

where $e_j$ denotes the $j$th column of an identity matrix. From (7a),

$$\begin{aligned} e_k^T \tilde{Z}_n e_j &= e_k Z_d^{-1/2} Z_n Z_d^{-1/2} e_j \\ &= (e_k^T Z_d^{-1/2} e_k)(e_k^T Z_n e_j)(e_j^T Z_d^{-1/2} e_j) \end{aligned} \tag{9}$$

It is recalled[4,5] that $\xi_{jj} = 2\zeta_j\omega_j$ so that

$$e_j^T Z_d^{-1/2} e_j = [(\omega_j^2 - \Omega^2) + i2\zeta_j\omega_j\Omega]^{-1/2} \tag{10}$$

and that

$$e_k^T Z_n e_j = i\Omega\xi_{kj} \tag{11}$$

where $\xi_{kj}$ is the $kj$th element of the modal damping matrix for $j \neq k$. The largest value of $e_k\tilde{Z}_n e_j$ is realized whenever $\Omega$ equals either $\omega_j$ or $\omega_k$. Without loss of generality, it may be assumed that $\Omega = \omega_j < \omega_k$. Then

$$e_j^T Z_d^{-1/2} e_j = (i2\zeta_j\omega_j^2)^{-1/2} = (i2\zeta_j\omega_j^2)^{-1}(i2\zeta_j\omega_j^2)^{1/2} \tag{12}$$

Furthermore,

$$i\omega_j\xi_{kj}(i2\zeta_j\omega_j^2)^{-1} = \xi_{kj}/2\zeta_j\omega_j = \xi_{kj}/\xi_{jj} \tag{13}$$

Substitution of (10) through (13) back into (9), with $\Omega = \omega_j$, leads to

$$e_k^T \tilde{Z}_n e_j = \left[\frac{i2\zeta_j\omega_j^2}{(\omega_k^2 - \omega_j^2) + i2\zeta_k\omega_j\omega_k}\right]^{1/2}\left(\frac{\xi_{kj}}{\xi_{jj}}\right)$$

Finally,

$$|e_k^T \tilde{Z}_n e_j| = \left\{\frac{2\zeta_j}{[(\beta^2-1)^2 + 4\zeta_k^2\beta^2]^{1/2}}\right\}^{1/2}\left|\frac{\xi_{kj}}{\xi_{jj}}\right| \tag{14}$$

where $\beta \equiv \omega_k/\omega_j > 1$. Neglecting the $\zeta_k^2$ term in (14), one finds that (8) is satisfied if

$$[2\zeta_j/(\beta^2-1)]^{1/2} << 1 \tag{15}$$

provided that $\xi_{kj}/\xi_{jj} \sim 1$ or smaller. For example, if $\zeta_j = 0.015$ and $\beta = 2$, the left-hand-side of (15) equals 0.1.

## Conclusions

It is concluded from this derivation that the degree of coupling between two classical normal modes depends not only on the ratio of the off-diagonal to diagonal terms of the modal damping matrix, but also on the percent of critical damping in the two modes and their frequency separation. The higher the percent of critical damping, the greater the frequency separation must be to uncouple the equations. Even when the classical normal modes do not diagonalize the damping matrix, the equations of motion will be uncoupled for all practical purposes, provided that adequate frequency separation exists between the modes.

## References

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# Stress Concentration in the Plastic Range

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## Nomenclature

| | | |
|---|---|---|
| $\epsilon$ | = | strain or strain at discontinuity |
| $\epsilon_p$ | = | strain away from discontinuity |
| $E$ | = | elastic modulus |
| $\bar{E}_s$, $E_s$, $E_p$ | = | secant moduli |
| $K_{el}$ | = | elastic stress concentration factor |
| $K_p$ | = | plastic stress concentration factor |
| $K_\epsilon$ | = | plastic strain concentration factor |
| $p$ | = | stress applied away from discontinuity |
| $p_e$ | = | "equivalent elastic" stress applied away from discontinuity |
| $\sigma$ | = | stress or stress at discontinuity |
| $\sigma_y$ | = | yield stress in simple tension |
| $\sigma_e$ | = | "equivalent elastic" stress at discontinuity |

## Introduction

STRUCTURAL components used in industry very often have discontinuities at which stress and strain

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